B-CONVEXITY AND REFLEXIVITY

BY

DANIEL P. GIESY

ABSTRACT

If $\varepsilon > 1/4$ and X is 3,ε-convex then X is reflexive. Some additional values of k and ε with $k \geqq 4$ are found for which k,ε-convexity implies reflexivity.

If $k \geqq 2$ and $\varepsilon > 0$, a Banach space X is called *k,ε-convex* if, for each choice of $x_1, \cdots, x_k$ in X with $\| x_i \| \leqq 1$, $1 \leqq i \leqq k$, then

$$\| \pm x_1 \pm x_2 \pm \cdots \pm x_k \| \leqq k(1-\varepsilon)$$

for some choice of the $+$ and $-$ signs. X is *B-convex* if, for some $k \geqq 2$ and $\varepsilon > 0$, X is k, ε-convex.

B-convexity was introduced by A. Beck [1] as a characterization of Banach spaces X for which the "uniformly bounded variances" Strong Law of Large Numbers is valid for X-valued random variables. A general study of B-convexity is given in [4].

B-convexity is a generalization of uniform convexity, long known to imply reflexivity, and so the question of whether or not all B-convex spaces are reflexive arose naturally. Several results support the conjecture that B-convexity implies reflexivity including results in [4] showing parallels in structure between the classes of B-convex and reflexive spaces, the theorem of R. C. James ([9], Theorem 1.1) that $2, \varepsilon$-convexity implies reflexivity, and the author's theorem, [5], that a B-convex normed Riesz space (Banach lattice) is reflexive.

In this paper, we show that if $k \geqq 3$ and $\varepsilon > 1 - 9/4k$, then all k, ε-convex spaces are reflexive (Theorem 3 and Corollary 6). Since (Theorem 7 and Remark 8)

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if $k \geqq 6$, $\varepsilon > 1 - 9/4k$, and X is k, ε-convex, then X is finite-dimensional, we see that the results of this paper are of interest only for $k = 3$, 4, and 5.

LEMMA 1. *If $2 \leqq n < k$, $k\varepsilon > k - n$, and X is k,ε-convex, then X is n, δ-convex with $\delta = n^{-1}(k\varepsilon - k + n)$.*

PROOF. If X is k,ε-convex, $x_1, \cdots, x_n \in X$ with $\|x_i\| \leqq 1$, and $x_{n+1} = \cdots = x_k = 0$, then for some choice of + and − signs

$$\| \pm x_1 \pm \cdots \pm x_k \| \leqq k(1 - \varepsilon)$$

from which it follows that for some choice of the + and − signs,

$$\| \pm x_1 \pm \cdots \pm x_n \| \leqq k(1 - \varepsilon) = n(1 - \delta).$$

REMARK 2. By Lemma 1, if X is $3, \varepsilon$-convex with $\varepsilon > \frac{1}{3}$, then X is $2, \delta$-convex with $\delta > 0$, hence by James' theorem [9, Theorem 1.1] X is reflexive. However, there do exist spaces which are $3, \varepsilon$-convex with $\varepsilon > \frac{1}{4}$ but which are not $2, \delta$-convex for any $\delta > 0$ (for a trivial example, 2-dimensional l_1 which is precisely $3, \frac{1}{3}$-convex) so that our present Theorem 3 is not included in James' theorem. Nor is James' theorem included in ours as is demonstrated by any l_p space of dimension at least 3 with p close enough to 1.

THEOREM 3. *If a Banach space X is $3, \varepsilon$-convex with $\varepsilon > \frac{1}{4}$, then X is reflexive.*

PROOF. Suppose X is not reflexive. For each $\varepsilon > \frac{1}{4}$, we will find $x', y', z' \in X$ with $\|x'\| \leqq 1$, $\|y'\| \leqq 1$, $\|z'\| \leqq 1$, and

$$\| \pm x' \pm y' \pm z' \| > 3(1 - \varepsilon)$$

for all choices of the + and − signs, so that X is not $3, \varepsilon$-convex. The construction we give of the sets $S(k_1, \cdots . k_{2n})$ and the numbers K_n follows James [10] which is a simplification of the argument used by James in [9] to establish reflexivity of $2, \varepsilon$-convex spaces for any $\varepsilon > 0$.

Since X is not reflexive, if $0 < \theta < 1$, then there exist $z_n \in X$ and $f_n \in X^*$ such that $\|z_n\| \leqq 1$ and $\|f_n\| \leqq 1$ for all n and $f_n(z_k) = \theta$ for $n \leqq k$, $f_n(z_k) = 0$ for $n > k$ (James [8, Theorem 7]).

Now for $1 \leqq k_1 < k_2 < \cdots < k_{2n}$, set

$$S(k_1, \cdots, k_{2n}) = \{x \in X : f_k(x) = (-1)^i \theta \text{ if } 1 \leqq i \leqq n \text{ and } k_{2i-1} \leqq k \leqq k_{2i}\}$$

and

$$K(k_1, \cdots, k_{2n}) = \inf\{\|x\| : x \in S(k_1, \cdots, k_{2n})\}.$$

Since

$$\sum_{i=1}^{n} (-1)^i (z_{k_{2i}} - z_{k_{2i-1}-1}) \in S(k_1, \cdots, k_{2n})$$

we have $K(k_1, \cdots, k_{2n}) \leqq 2n$. Now let

$$K_n = \liminf_{k_1 \to \infty} \left(\liminf_{k_2 \to \infty} \left(\cdots \liminf_{k_{2n} \to \infty} K(k_1, \cdots, k_{2n}) \cdots \right)\right).$$

Then $K_n \leqq 2n$. Since we also have

$$K(k_1, \cdots, k_{2n}) \leqq K(k_1, \cdots, k_{2n+2})$$

it follows that $K_n \leqq K_{n+1}$. Also $\theta \leqq K(k_1, \cdots, k_{2n})$ so $\theta \leqq K_n$. We now need the following lemma concerning the growth of sequences like $\{K_n\}$.

LEMMA 4. *If $\{a_n\}$ is a non-decreasing sequence of positive real numbers for which $a_n = \mathcal{O}(n)$ and $\{j_n\}$ and $\{k_n\}$ are increasing sequences of integers for which $j_{n+1} - j_n = \mathcal{O}(1)$ and $\liminf_{n\to\infty} j_n/k_n = r$ with $0 < r < 1$, then $\limsup_{n\to\infty} a_{j_n}/a_{k_n} \geqq r$.*

PROOF. Suppose $\limsup_{n\to\infty} a_{j_n}/a_{k_n} < r$. Then there are numbers N, s, and t with $0 < s < t < r < 1$ such that for $n \geqq N$, $a_{j_n}/a_{k_n} \leqq s$ and $j_n/k_n \geqq t$. Define sequences $\{l_n\}$ and $\{m_n\}$ of integers by requiring that $l_1 = j_N$ and, for each n, if $l_n = j_i$, then $m_n = k_i$ and $l_{n+1} = j_v$ where $j_{v-1} < k_i \leqq j_v$. Then $a_{l_n} \leqq s a_{m_n} \leqq s a_{l_{n+1}}$ so $a_{l_n} \geqq s^{1-n} a_{l_1}$. On the other hand, if $j_{n+1} - j_n \leqq M$ for all n, then $l_n \geqq t m_n \geqq t(l_{n+1} - M)$ so

$$\begin{aligned} l_n &\leqq t^{1-n} l_1 + M(t^{2-n} + t^{3-n} + \cdots + 1) \\ &= t^{1-n}[l_1 + M(t - t^{n-1})/(1-t)]. \end{aligned}$$

Hence

$$a_{l_n}/l_n \geqq (t/s)^{n-1} \cdot a_{l_1}/[l_1 + M(t - t^{n-1})/(1-t)].$$

Since $t < 1$, the bracketed portion is bounded as $n \to \infty$. Also $s < t$, so we have a contradiction to the assumption that $a_n = \mathcal{O}(n)$.

It follows from Lemma 4 that

$$\limsup_{n\to\infty} \frac{K_{6n-3}}{K_{8n}} \geqq \frac{3}{4}.$$

Now fix $\delta > 0$. Pick n for which $K_{6n-3}/K_{8n} \geqq 3(1-\delta)/4$. Now if integers $p_1, \cdots, p_{16n}; q_1, \cdots, q_{16n-2}; r_1, \cdots, r_{16n-4}$ are chosen so that the sequence

$$p_1, q_1, r_1; p_2, p_3, q_2, q_3, r_2, r_3, p_4, p_5,$$
$$q_4, q_5, r_4, r_5, \cdots, p_{4n-2}, p_{4n-1}, q_{4n-2},$$
$$q_{4n-1}, r_{4n-2}, r_{4n-1}; p_{4n}, p_{4n+1}, q_{4n},$$
$$q_{4n+1}; p_{4n+2}, p_{4n+3}, q_{4n+2}, q_{4n+3}, r_{4n},$$
$$r_{4n+1}, \cdots, p_{8n-2}, p_{8n-1}, q_{8n-2}, q_{8n-1},$$
$$r_{8n-4}, r_{8n-3}; p_{8n}, p_{8n+1}, r_{8n-2}, r_{8n-1};$$
$$p_{8n+2}, p_{8n+3}, q_{8n}, q_{8n+1}, r_{8n}, r_{8n+1}, \cdots,$$
$$p_{12n-2}, p_{12n-1}, q_{12n-4}, q_{12n-3}, r_{12n-4}, r_{12n-3};$$
$$p_{12n}, p_{12n+1}, q_{12n-2}, q_{12n-1}; p_{12n+2}$$
$$p_{12n+3}, q_{12n}, q_{12n+1}, r_{12n-2}, r_{12n-1}, \cdots,$$
$$p_{16n-2}, p_{16n-1}, q_{16n-4}, q_{16n-3}, r_{16n-6}, r_{16n-5};$$
$$p_{16n}, q_{16n-2}, r_{16n-4}$$

is increasing and if

$$\begin{aligned} &x \in S(p_1, \cdots, p_{16n}), \\ (1) \qquad &y \in S(q_1, \cdots, q_{16n-2}), \text{ and} \\ &z \in S(r_1, \cdots, r_{16n-4}), \text{ then} \end{aligned}$$

$$(\tfrac{1}{3})(x + y + z) \in S(r_1, p_2, r_3, p_4, \cdots, r_{4n-1}, p_{4n};$$
$$q_{4n+5}, r_{4n+2}, q_{4n+7}, r_{4n+4}, \cdots, q_{8n-1}, r_{8n-4};$$
$$p_{8n+5}, q_{8n+2}, p_{8n+7}, q_{8n+4}, \cdots, p_{12n+1}, q_{12n-2}),$$
$$(\tfrac{1}{3})(-x - y + z) \in S(q_3, r_2, q_5, r_4, \cdots, q_{4n-1}, r_{4n-2};$$
$$q_{4n+1}, p_{4n+2}; r_{4n+1}, p_{4n+4}, r_{4n+3}, p_{4n+6},$$
$$\cdots, r_{8n-3}, p_{8n}; p_{12n+5}, q_{12n+2}, p_{12n+7},$$
$$q_{12n+4}, \cdots, p_{16n-1}, q_{16n-4}),$$
$$(\tfrac{1}{3})(-x + y - z) \in S(p_{4n+3}, q_{4n+2}, p_{4n+5}, q_{4n+4},$$
$$\cdots, p_{8n-1}, q_{8n-2}; p_{8n+1}, r_{8n-2}; q_{8n+1},$$
$$r_{8n}, q_{8n+3}, r_{8n+2}, \cdots, q_{12n-3}, r_{12n-4};$$
$$q_{12n-1}, p_{12n+2}; r_{12n-1}, p_{12n+4}, r_{12n+1},$$
$$p_{12n+6}, \cdots, r_{16n-5}, p_{16n}),$$

and

$$(\tfrac{1}{3})(-x+y+z)\in S(p_3, q_2, p_5, q_4, \cdots, p_{4n+1}, q_{4n}; r_{8n+1}, p_{8n+4}, r_{8n+3}, p_{8n+6}, \cdots, r_{12n-3}, p_{12n}; q_{12n+3}, r_{12n}, q_{12n+5}, r_{12n+2}, \cdots, q_{16n-3}, r_{16n-6}).$$

From the definition of $\{K_n\}$ we see that it is possible to choose the p's, q's, and r's so that there exist x, y, and z satisfying (1) and, in addition, both

$$\|x\| \leqq K_{8n}(1+\delta),$$
$$\|y\| \leqq K_{8n-1}(1+\delta) \leqq K_{8n}(1+\delta),$$
$$\|z\| \leqq K_{8n-2}(1+\delta) \leqq K_{8n}(1+\delta),$$

and

$$\|+x+y+z\| \geqq 3K_{6n-3}(1-\delta),$$
$$\|-x-y+z\| \geqq 3K_{6n-3}(1-\delta),$$

$$\|-x+y-z\| \geqq 3K_{6n-1}(1-\delta) \geqq 3K_{6n-3}(1-\delta),$$
$$\|-x+y+z\| \geqq 3K_{6n-3}(1-\delta).$$

So if $c = [K_{8n}(1+\delta)]^{-1}$, $x' = cx$, $y' = cy$, and $z' = cz$, then $\|x'\| \leqq 1$, $\|y'\| \leqq 1$, and $\|z'\| \leqq 1$, while for each choice of the $+$ and $-$ signs,

$$\|\pm x' \pm y' \pm z'\| \geqq 3\,\frac{K_{6n-3}(1-\delta)}{K_{8n}(1+\delta)} \geqq 3\cdot\frac{3}{4}\cdot\frac{(1-\delta)^2}{(1+\delta)}.$$

Since $\delta > 0$ is arbitrary, it follows that for every $\varepsilon > \frac{1}{4}$, X is not $3,\varepsilon$-convex.

Remark 5. The author conjectures with a high degree of confidence that the technique of proof of Theorem 3 can be extended to show that if $\varepsilon > 1 - k2^{1-k}$ and X is k,ε-convex, then X is reflexive. However, in view of the easily established inequality $1 - k2^{1-k} > 1 - 9/(4k)$ for $k \geqq 4$ and the following corollary, this seems pointless.

Corollary 6. *If* $k \geqq 3$, $\varepsilon > 1 - 9/(4k)$, *and* X *is* k,ε-*convex, then* X *is reflexive.*

PROOF. From Lemma 1, it follows that X is $3,\delta$-convex with $\delta > \frac{1}{4}$, so that Theorem 3 applies.

THEOREM 7. *If X is an infinite dimensional k,ε-convex space, then* $\varepsilon \leqq 1 - k^{-\frac{1}{2}}$.

PROOF. Since by a theorem of Dvoretzky (see, e.g. [3]) X contains subspaces approximating k-dimensional Hilbert space as closely as one wants, it is sufficient to establish the inequality for k-dimensional Hilbert space. But there, it is the immediate consequence of the fact that, if $x_1, \cdots, x_k$ is an orthonormal sequence in a Hilbert space, then for all choices of the signs,

$$\| \pm x_1 \pm x_2 \pm \cdots \pm x_k \| = k^{\frac{1}{2}}.$$

REMARK 8. From the easily established k-parallelpiped law in Hilbert space

$$\Sigma \; \| \pm x_1 \pm \cdots \pm x_k \|^2 = 2^k \sum_{i=1}^{k} \| x_i \|^2$$

where the first Σ is over the 2^k choices of signs, we see that any Hilbert space (or even inner product space) is k, $1 - k^{-\frac{1}{2}}$-convex so that the bound in Theorem 7 is sharp.

If we say that every space is $k,0$-convex and for fixed X let $\varepsilon_k = \sup\{\varepsilon \geqq 0: X$ is k,ε-convex$\}$ (so that, in fact, X is k, ε_k-convex), then (in a different notation) [4, Lemma I.4, p. 119] tells us that either $\varepsilon_k \equiv 0$ in which case X is not B-convex or $\varepsilon_k \to 1$ in which case X is B-convex and in the latter case there exists a $\gamma > 0$ for which $1 - \varepsilon_k = \mathcal{O}(k^{-\gamma})$. It follows from Theorem 7 that if X is infinite dimensional, $\gamma \leqq \frac{1}{2}$. By a routine extension of the technique of [4, example I.3(i), p. 118] (if $k > n$ and $x_1, \cdots, x_k$ are vectors in n-space, then there are scalars $a_1, \cdots, a_k$ with all $|a_i| \leqq 1$, $k - n$ of the $|a_i| = 1$, and $\Sigma_1^k \, a_i x_i = 0$), we see that for $k > n$, all n-dimensional spaces are k, $1 - n/k$-convex, so in this case $\gamma = 1$.

Since $1 - k^{-\frac{1}{2}} < 1 - 9/(4k)$ for $k \geqq 6$, we see that Corollary 6 can only be applicable to an infinite dimensional X for $k = 3$, 4, or 5. Since every finite dimensional space is reflexive, these are the only interesting cases of Corollary 6. Summarizing James' theorem and the present results, a k,ε-convex space is reflexive in case

$$\begin{aligned} k &= 2 \text{ and } \varepsilon > 0, \\ k &= 3 \text{ and } \varepsilon > 1/4, \\ k &= 4 \text{ and } \varepsilon > 7/16, \\ k &= 5 \text{ and } \varepsilon > 11/20. \end{aligned} \tag{2}$$

COROLLARY 9. *If X is k, ε-convex with k and ε as in* (2) *above, then X is super-reflexive and stable.*

PROOF. Super-reflexivity follows from [7, Theorem 3] (see [7] for the definition of super-reflexivity) and stability follows from Brunel and Sucheston [2, Theorem 1] or [6, Corollary 1]. (Stability is defined in both of these papers.)

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DEPARTMENT OF MATHEMATICS
WESTERN MICHIGAN UNIVERSITY
KALAMAZOO MICHIGAN, U. S. A.